EVALUATING THE HEAT PROOFING PROPERTIES OF ANISOTROPIC INSULATION

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The thermal resistance of vacuum-shield insulation is evaluated on the basis of earlier test data [1] on the anisotropy of heat conduction through it.

The data in [1] indicate a very pronounced anisotropy of heat conduction through a vacuum-shield thermal insulation along and across its layers. For this reason, an evaluation of the thermal resistance of such an insulation must take into account this anisotropy as well as the specific structural conditions under which it is used.

We will consider the case of steady-state heat transmission through a flat layer of anisotropic vacuum-shield insulation with heat transfer at its end surfaces. This case is typical of many practical structural designs involving the installation of vacuum-shield insulation.

The schematic diagram in Fig. 1 shows the transverse section through a flat layer of anisotropic insulation having the shape of an infinitely long prism. The heat transfer at the end surfaces is defined by boundary conditions of the third kind.

Let the constant thermal conductivities along the x- and y-axis be λ_x and λ_y, respectively, and the temperature of the insulation be T. The differential equation of heat conduction in this case will be

$$\lambda_x \frac{\partial^2 T}{\partial x^2} + \lambda_y \frac{\partial^2 T}{\partial y^2} = 0. \tag{1}$$

and the boundary conditions:

$$\text{for } y = b \;\; \lambda_y \frac{\partial T}{\partial y} = \alpha_1 (T_1 - T);$$

$$\text{for } y = 0 \;\; \lambda_y \frac{\partial T}{\partial y} = \alpha_2 (T - T_2);$$

$$\text{for } x = 0 \;\; \lambda_x \frac{\partial T}{\partial x} = \alpha_1 (T - T_1);$$

$$\text{for } x = a \;\; \lambda_x \frac{\partial T}{\partial x} = \alpha_2 (T_2 - T).$$

In the dimensionless coordinates

$$\xi = \frac{x}{b} \sqrt{\frac{\lambda_y}{\lambda_x}} \text{ and } \eta = \frac{y}{b},$$

Eq. (1) becomes

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} = 0. \tag{2}$$

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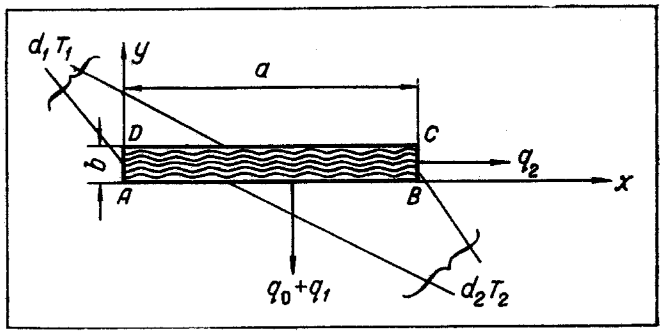

Fig. 1. Schematic diagram for calculating the heat transfer in laminated insulation.

We introduce the following notation:

$$\frac{\lambda_x}{\lambda_y} \equiv k; \quad R_y = \frac{b}{\lambda_y}; \quad R_1 = \frac{1}{\alpha_1}; \quad R_2 = \frac{1}{\alpha_2}.$$

Then the boundary conditions can be rewritten as

$$\text{for } \eta = 1 \quad \frac{\partial T}{\partial \eta} = \frac{R_y}{R_1}(T_1 - T);$$

$$\text{for } \eta = 0 \quad \frac{\partial T}{\partial \eta} = \frac{R_y}{R_2}(T - T_2);$$

$$\text{for } \xi = 0 \quad \frac{\partial T}{\partial \xi} = \frac{R_y}{R_1}\cdot\frac{1}{\sqrt{k}}(T - T_1);$$

$$\text{for } \xi = \frac{a}{b\sqrt{k}} \quad \frac{\partial T}{\partial \xi} = \frac{R_y}{R_2}\cdot\frac{1}{\sqrt{k}}(T_2 - T).$$

The solution will be sought in the form

$$T = T_0 + f(\xi, \eta),$$

with T_0 corresponding to a one-dimensional temperature field (disregarding the heat transfer at the end surface of the specimen).

If we let $T_0 = C_0 + C_1\eta$, then the boundary conditions will yield

$$C_1 = \frac{R_y}{R_1}(T_1 - C_0 - C_1); \quad C_1 = \frac{R_y}{R_2}(C_0 - T_2),$$

from where

$$C_0 - T_2 = \frac{R_2}{R_1}(T_1 - C_0) - \frac{R_y}{R_1}(C_0 - T_2).$$

We add the following notation:

$$C_0 - T_2 \equiv C_0'; \quad T_1 - T_2 \equiv \Delta T,$$

so that

$$C_0' = \frac{R_2 \Delta T}{R_y + R_1 + R_2}; \quad C_1 = \frac{R_y \Delta T}{R_y + R_1 + R_2}.$$

Now Eq. (2) becomes

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0 \tag{3}$$

and the boundary conditions:

$$\text{for } \eta = 1 \quad \frac{\partial f}{\partial \eta} = -\frac{R_y}{R_2} f; \tag{4}$$

$$\text{for } \eta=0 \quad \frac{\partial f}{\partial \eta}=\frac{R_y}{R_2} f; \tag{5}$$

$$\text{for } \xi=0 \quad \frac{\partial f}{\partial \xi}=\frac{R_y}{R_1 \sqrt{k}}\left(f+C_0'+C_1 \eta-\Delta T\right); \tag{6}$$

$$\text{for } \xi=\frac{a}{b \sqrt{k}} \quad \frac{\partial f}{\partial \xi}=-\frac{R_y}{R_2 \sqrt{k}}\left(f+C_0'+C_1 \eta\right). \tag{7}$$

We will seek function f which is harmonic within the rectangle ABCD:

$$f=\sum_{n=1}^{\infty}\left[\left(A_n' \sin \mu_n \eta+B_n' \cos \mu_n \eta\right) \operatorname{ch} \mu_n \xi+\left(A_n'' \sin \mu_n \eta+B_n'' \cos \mu_n \eta\right) \operatorname{sh} \mu_n \xi\right], \tag{8}$$

with μ_n defined by the transcendental equation

$$\operatorname{ctg} \mu_n=\frac{R_1}{R_1+R_2}\left(\frac{R_2}{R_y} \mu_n-\frac{R_y}{R_1} \cdot \frac{1}{\mu_n}\right), \tag{9}$$

which follows from a simultaneous solution of (4), (5), and (8). From the same equations we also obtain

$$\frac{B_n'}{A_n'}=\frac{B_n''}{A_n''}=\frac{R_2}{R_y} \mu_n. \tag{10}$$

On the basis of (8) and (10), we can express f as

$$f=\sum_{n=1}^{\infty}\left(A_n' \operatorname{ch} \mu_n \xi+A_n'' \operatorname{sh} \mu_n \xi\right) S_n(\eta), \tag{11}$$

where

$$S_n(\eta)=\sin \mu_n \eta+\frac{R_2}{R_y} \mu_n \cos \mu_n \eta. \tag{12}$$

Inserting solution (11) into the boundary conditions (6) and (7), we find

$$\sum_{n=1}^{\infty}\left(\mu_n A_n''-\frac{R_y}{R_1 \sqrt{k}} A_n'\right) S_n(\eta)=\frac{R_y}{R_1 \sqrt{k}}\left(C_0'-\Delta T+C_1 \eta\right), \tag{13}$$

$$\sum_{n=1}^{\infty}\left[\mu_n\left(A_n' \operatorname{sh} \mu_n \frac{a}{b \sqrt{k}}+A_n'' \operatorname{ch} \mu_n \frac{a}{b \sqrt{k}}\right)\right.$$

$$\left.+\frac{R_y}{R_2 \sqrt{k}}\left(A_n' \operatorname{ch} \mu_n \frac{a}{b \sqrt{k}}+A_n'' \operatorname{sh} \mu_n \frac{a}{b \sqrt{k}}\right)\right] S_n(\eta)=-\frac{R_y}{R_2} \cdot \frac{1}{\sqrt{k}}\left(C_0'+C_1 \eta\right). \tag{14}$$

In order to determine coefficients A_n' and A_n'', we expand the right-hand sides of Eqs. (13) and (14) into series in terms of eigenfunctions $S_n(\eta)$ of the Sturm–Liouville boundary-value problem corresponding to the boundary conditions (4) and (5).

Finally, we have

$$A_n''=\frac{1}{\mu_n} \cdot \frac{R_y}{R_1} \cdot \frac{1}{\sqrt{k}} A_n'+\frac{1}{\mu_n \sqrt{k}} \cdot \frac{R_y}{R_1}\left[\left(C_0'-\Delta T\right) \gamma_n+C_1 \beta_n\right], \tag{15}$$

where

$$\gamma_n=\frac{S_n}{S_{nn}}; \quad \beta_n=\frac{S_{n\eta}}{S_{nn}};$$

$$S_n=\frac{1-\cos \mu_n}{\mu_n}+\frac{R_2}{R_y} \sin \mu_n;$$

$$S_{n\eta} = \frac{\sin\mu_n}{\mu_n^2} - \frac{\cos\mu_n}{\mu_n} + \frac{R_2}{R_y}\left(\sin\mu_n - \frac{1-\cos\mu_n}{\mu_n}\right);$$

$$S_{nn} = \frac{1}{2}\left(1+\mu_n^2\frac{R_2^2}{R_y^2}\right) - \left(1-\mu_n^2\frac{R_2^2}{R_y^2}\right)\frac{\sin 2\mu_n}{4\mu_n} + \frac{R_2}{R_y}\frac{(1-\cos 2\mu_n)}{2}; \tag{16}$$

$$A_n' = -\frac{\Lambda_n}{\Pi_n},$$

where

$$\Lambda_n = \frac{R_y}{\sqrt{k}R_2}(C_0'\gamma_n + C_1\beta_n) + \frac{1}{\mu_n\sqrt{k}}\cdot\frac{R_y}{R_1}\left[\mu_n \operatorname{ch}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right)\right.$$
$$\left.+\frac{R_y}{R_2}\cdot\frac{1}{\sqrt{k}}\operatorname{sh}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right)\right]\left[(C_0'-\Delta T)\gamma_n + C_1\beta_n\right];$$

$$\Pi_n = \left(\mu_n + \frac{1}{\mu_n k}\cdot\frac{R_y^2}{R_1R_2}\right)\operatorname{sh}\left(\frac{\mu_n a}{\sqrt{k}b}\right) + \frac{R_y}{R_2\sqrt{k}}\left(1+\frac{R_2}{R_1}\right)\operatorname{ch}\left(\frac{\mu_n a}{\sqrt{k}b}\right).$$

The thermal flux density on a strip of unit width along segment AB ($y = 0$) is $q = \alpha_2[T(\xi, 0) - T_2]$. We will express it as a sum $q = q_0 + q_1$ of the thermal flux corresponding to a one-dimensional temperature field

$$q_0 = \frac{\Delta T}{R_y + R_1 + R_2},$$

and an additional thermal flux due to heat transfer at the end surfaces

$$q_1 = \frac{f(\xi, 0)}{R_2}$$

or

$$q_1 = \frac{1}{R_y}\sum_{n=1}^{\infty}(A_n' \operatorname{ch}\mu_n\xi + A_n'' \operatorname{sh}\mu_n\xi)\,\mu_n.$$

The total thermal flux on a strip of unit width along AB ($y = 0$) is

$$Q = \int_0^a q\,dx = b\sqrt{k}\int_0^{\frac{a}{b\sqrt{k}}} q\,d\xi = Q_0 + Q_1, \tag{17}$$

where $Q_0 = q_0 a$ is the total thermal flux on a strip of unit width in a one-dimensional temperature field,

$$Q_1 = \frac{\Delta T\sqrt{k}}{R_y}\sum_{n=1}^{\infty}\left\{A_n' \operatorname{sh}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right) + A_n''\left[\operatorname{ch}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right) - 1\right]\right\}, \tag{18}$$

and Q_1 is the additional thermal flux due to heat transfer at the end surfaces.

Analogously,

$$Q_2 = -\frac{b\sqrt{k}\Delta T}{R_y}\sum_{n=1}^{\infty}\mu_n s_n\left[A_n' \operatorname{sh}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right) + A_n'' \operatorname{ch}\left(\frac{\mu_n}{\sqrt{k}}\cdot\frac{a}{b}\right)\right], \tag{19}$$

where Q_2 is the total thermal flux on a strip of unit width along BC (at the $x = a$-end).

Of practical interest is the edge effect Q^* in an insulation with an infinitely long x-dimension (i.e., when $a/b\sqrt{k} \to \infty$). After a few transformations, we have

$$Q^* = \frac{b\sqrt{k}\Delta T}{R_y + R_1 + R_2}\sum_{n=1}^{\infty}\frac{(R_y + R_1)\gamma_n - R_y\beta_n}{R_y + \mu_n\sqrt{k}R_1}. \tag{20}$$

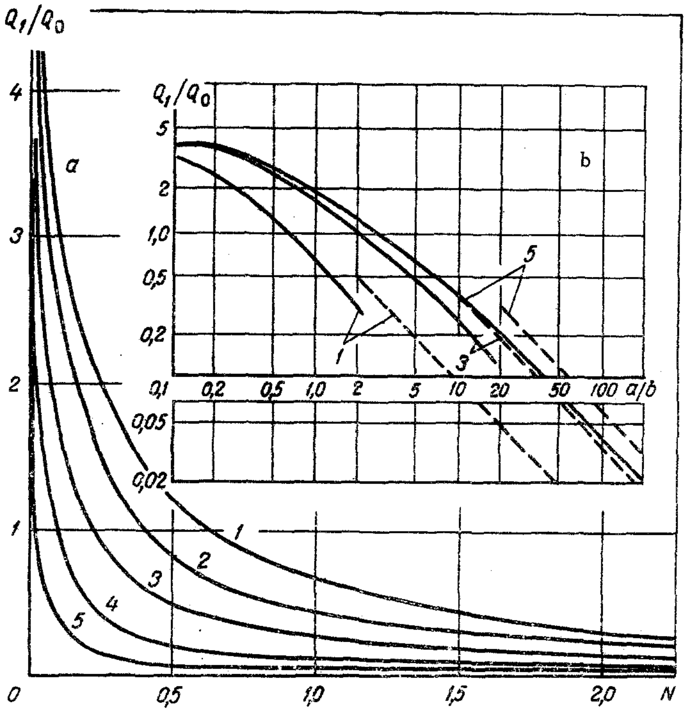

Fig. 2. Heat transmission through the insulation, as a function of the insulation parameters. a: Bi = 20, Bi' = 10; 1) k = 1; 2) 10; 3) 100; 4) 1000; 5) 10,000; b: 1) k = 1; 3) 100; 5) 10,000. Solid curves calculated according to formula (18); dashed curves calculated according to formula (20).

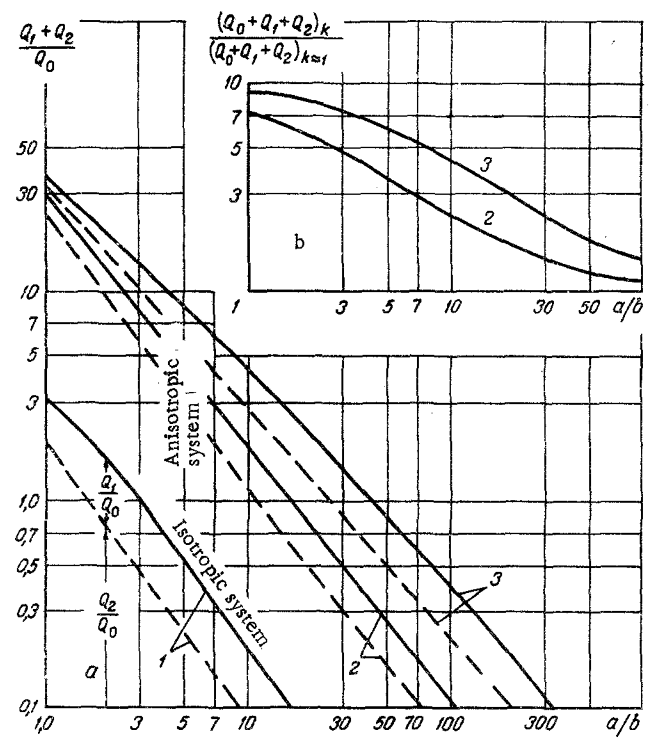

Fig. 3. Heat transmission through the insulation, as a function of the insulation parameters, with Bi = 100 and Bi' = 50. a: 1) k = 1; 2) 100; 3) 10,000. Solid lines represent $(Q_1 + Q_2)/Q_0$; dashed lines represent Q_2/Q_0; b: 2) k = 100; 3) 10,000.

The analytical solutions (18), (19), and (20) are shown graphically in Figs. 2 and 3, where the results of calculations on a digital computer are given in critical form. The graphs represent relations for the referred thermal flux $\overline{Q}$ in terms of Q_1/Q_0, $(Q_1 + Q_2)/Q_0$, $(Q_1 + Q_2 + Q_0)_k/(Q_1 + Q_2 + Q_0)_{k=1}$ as functions of $k = \lambda_x/\lambda_y$, $N = a/b\sqrt{k}$ or a/b, at constant values of $Bi = R_y/R_1$ and $Bi' = R_y/R_2$.

The dimensionless ratio Q_1/Q_0 characterizes the effect of heat transfer at the end surfaces on the thermal flux on the insulation surface along the segment AB, Q_2/Q_0 characterizes the thermal flux on the end surfaces of the insulation (segment BC), $(Q_1 + Q_2)/Q_0$ characterizes the effect of heat transfer at the end surfaces on the total heat transmitted through the insulation, and $(Q_0 + Q_1 + Q_2)_k/(Q_0 + Q_1 + Q_2)_{k=1}$ characterizes the effect of the anisotropy of the insulation on the heat transmitted through the insulation.

The calculated results indicate that the additional thermal fluxes at the end surfaces of an anisotropic insulation may reduce the insulation effectiveness to a fraction. This, in turn, indicates the need for special designs which would limit the heat transfer at the end surfaces when anisotropic laminated thermal insulation is used.

NOTATION

T	is the temperature;
T_1, T_2	are the ambient temperatures at the boundaries of the insulation layer;
x, y	are the coordinates;
a, b	are the linear dimensions of the insulation layer;
α_1, α_2	are the coefficients of heat transfer at the boundaries of the insulation layer;
q_0, q_1, q_2	are the thermal flux densities on the inner boundaries (AB and BC) of the insulation layer.

LITERATURE CITED

1. D. P. Lebedev, E. K. Zlobin, and V. V. Alekseev, Inzh.-Fiz. Zh., 21, No. 15 (1971).